Companions of MTL logics for reasoning about acceptable and tolerable propositions

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1 Introduction

Systems of mathematical fuzzy logic are formal systems designed to handle *graded truth*, making them particularly suitable for reasoning with imprecise or vague information. Their defining feature is the interpretation of formulas over a linearly ordered scale of truth values, typically within the real unit interval $[0, 1]$. This framework is especially well-suited for capturing the gradual nature of vagueness. A significant body of work on fuzzy logics has been developed within the framework of mathematical fuzzy logic (MFL) [2, 1].

In MFL, deductive systems have been traditionally defined based on the principle of *full truth-preservation*, where a formula is considered a consequence of a set of premises if, under every algebraic evaluation that assigns the truth value 1 (absolute truth) to all premises, the conclusion also receives the value 1. This generalizes the classical notion of logical consequence.

In this paper we consider yet a different type of companion logics, namely logics that preserve the notion of acceptability. We call a formula φ *acceptable*, for a given evaluation e, when φ is more true than false, that is, when $e(\varphi) > e(\neg\varphi)$. We also consider the dual notion, a formula φ is called *tolerable* when $\neg\varphi$ is not acceptable, i.e. when $e(\neg\neg\varphi) \geq e(\neg\varphi)$. We study the properties of these *acceptability* and *tolerance-preserving* companions and their relationship to their truth-preserving companions, as well as their axiomatisation in some particular cases. Throughout this paper we assume the MTL logics L we deal with are standard complete, that is, they are complete with respect to a family of standard L-algebras.

This short paper is a summary of part of the recently published article [4].

2 Logics of 'acceptance' and 'tolerance'

The most distinctive feature of any system of fuzzy logic is that the notion of truth (and falsity) is a matter of degree. In a sense, in each interpretation e of a logic L, one can consider that every proposition φ is assigned a degree of being true, represented by the value $e(\varphi)$ and a degree of being false, represented by

the value $e(\neg\varphi)$. Nonetheless, notice that the situation may be non-symmetric if the logic L is not involutive, that is, the degree in which $\neg\varphi$ is false, $e(\neg\neg\varphi)$, may be different from (bigger than, in fact) the degree in which φ is true, i.e. $e(\neg\neg\varphi) > e(\varphi)$.

Although the context is somewhat different, we take inspiration from [3] and consider that a proposition can be *accepted* (as if it were true) when it is "more true than false". In other words, we propose to define that a proposition φ is *accepted* by a given interpretation e whenever $e(\varphi) > e(\neg\varphi)$, or conversely, to say that the interpretation e accepts φ. Then, the question we pose ourselves is which is the logic that captures reasoning with accepted propositions.

Definition 1. *Let L be a finitary expansion of MTL which is complete with respect to a class $\mathcal{C}_L$ of standard L-algebras on $[0,1]$. Further let $[\mathbf{0},\mathbf{1}]_* \in \mathcal{C}_L$ be a standard L-algebra and let e be a $[\mathbf{0},\mathbf{1}]_*$-interpretation. We say that e* accepts *φ, written $e \models^{acc} \varphi$, whenever $e(\varphi) > e(\neg\varphi)$. The corresponding notion of consequence relation, denoted $\models^{acc}_L$, is accordingly defined as follows:*

$$\Gamma \models^{acc}_L \varphi \quad \text{if,} \quad \begin{array}{l} \text{for any } [\mathbf{0},\mathbf{1}]_* \in \mathcal{C}_L \text{ and any } [0,1]_*\text{-evaluation } e, \\ \text{if } e \models^{acc} \psi \text{ for any } \psi \in \Gamma, \text{ then } e \models^{acc} \varphi \text{ as well.} \end{array}$$

The following are some interesting properties of $\models^{acc}_L$ for any L being an extension of MTL:

Lemma 1. *The following properties of $\models^{acc}_L$ hold:*

1. *Consistency: $\models^{acc}_L \top$, $\not\models^{acc}_L \bot$*
2. *Paracompleteness: $\not\models^{acc}_L \varphi \vee \neg\varphi$, unless L has not a negation fixpoint.*
3. *Explosiveness: $\varphi, \neg\varphi \models^{acc}_L \bot$*
4. *Closure under conjunction:*
   (i) $\varphi, \psi \models^{acc}_L \varphi \wedge \psi$
   (ii) $\varphi, \psi \models^{acc}_L \chi$ iff $\varphi \wedge \psi \models^{acc}_L \chi$
5. *If $\models_L \varphi \to \psi$ then $\varphi \models^{acc}_L \psi$.*

Note that the above Item 5 admits an equivalent interesting reading, namely that: the restricted Modus Ponens rule:

$$\frac{\varphi, \quad \vdash_L \varphi \to \psi}{\psi}$$

is sound for $\models^{acc}_L$. Moreover, $\models^{acc}_L$ is interpretable in $\models_L$, indeed the following equivalence holds:

$$\varphi \models^{acc}_L \psi \quad \text{iff} \quad \psi \to \neg\psi \models_L \varphi \to \neg\varphi \tag{1}$$

It is also very natural to consider for a proposition φ a dual notion of being acceptable: we define φ is *tolerated* when $\neg\varphi$ is not acceptable. In other words, in the setting of a core fuzzy logic L, we can say that an L-evaluation

e tolerates φ when $e(\neg\varphi) \leq e(\neg\neg\varphi)$.[1] This leads to the following notion of tolerance-preserving logic.

Definition 2. *$\varphi \models_L^{tol} \psi$ if, for any L-evaluation e, if $e(\neg\neg\varphi) \geq e(\neg\varphi)$ then $e(\neg\neg\psi) \geq e(\neg\psi)$.*

It is very easy to check that $\models^{tol}$ is interpretable in both the acceptance logic $\models^{acc}$ and the logic L itself. Indeed, the following equivalences hold:

(i) $\varphi \models_L^{tol} \psi$ iff $\neg\psi \models_L^{acc} \neg\varphi$
(ii) $\Gamma \models_L^{tol} \psi$ iff $\{\neg\varphi \to \neg\neg\varphi \mid \varphi \in \Gamma\} \models_L \neg\psi \to \neg\neg\psi$

The logic $\models_L^{tol}$ has similar properties as $\models_L^{acc}$, but there is a main difference: while $\models_L^{acc}$ is not paraconsistent (it is explosive), $\models_L^{tol}$ is paraconsistent.

Lemma 2. *The following properties of $\models_L^{tol}$ hold:*

1. *Consistency: $\models_L^{tol} \top$, $\not\models_L^{tol} \bot$*
2. *Excluded middle: $\models_L^{tol} \varphi \vee \neg\varphi$*
3. *Paraconsistency: $\varphi, \neg\varphi \not\models_L^{tol} \bot$, unless L has not a negation fixpoint*
4. *Closure under conjunction:*
   (i) $\varphi, \psi \models_L^{tol} \varphi \wedge \psi$
   (ii) $\varphi, \psi \models_L^{tol} \chi$ iff $\varphi \wedge \psi \models_L^{tol} \chi$
5. *if $\models_L \varphi \to \psi$ then $\varphi \models_L^{tol} \psi$. (i.e. $\models_L^{\leq} \subseteq \models_L^{tol}$)*

It also follows that, if $\neg$ has a fixpoint, then $\models_L^{tol}$ is the paraconsistent (and not paracomplete) version of $\models_L^{acc}$, and vice versa, $\models_L^{acc}$ is the paracomplete (and not paraconsistent) version of $\models_L^{tol}$.

3 Axiomatisation of the acceptance and tolerance logic companions of a Δ-core fuzzy logic

The axiomatization of $\models_L^{acc}$ is rather simple in case the Δ connective is available. Indeed, let L be a Δ-core fuzzy logic such that the Baaz-Monteiro Δ operator is either primitive or definable, for instance MTL_Δ (primitive) or Gödel logic with involution $\mathrm{G}_\sim$ (definable). We further assume that L only has Modus Ponens and necessitation for Δ as inference rules, and that is complete with respect to a corresponding class of standard algebras $\mathcal{C}_L$.

In this logic we introduce the following definable connective:

$$\boxdot\varphi := \Delta(\neg\varphi \to \varphi) \wedge \neg\Delta(\varphi \to \neg\varphi).$$

Note that, for any standard interpretation e, we have (i) $e(\boxdot\varphi) = 1$ iff $e(\neg\varphi) < e(\varphi)$, and (ii) $e(\Delta\varphi) > e(\neg\Delta(\varphi))$ iff $e(\varphi) = 1$. It then follows that the logic $\models_L^{acc}$ is mutually interpretable in $\models_L$ itself; that is, we have:

[1] Notice that this condition is somewhat weaker than the condition of requiring $e(\neg\varphi) \leq e(\varphi)$, that may also be a reasonable alternative in MTL logics that are non IMTL logics.

$$\Gamma \models^{acc}_{L} \psi \quad \text{iff} \quad \boxdot\Gamma \models_{L} \boxdot\psi,$$
$$\Gamma \models_{L} \psi \quad \text{iff} \quad \Delta\Gamma \models^{acc}_{L} \Delta\psi.$$

where $\odot\Gamma$ is a shorthand for $\{\odot\varphi \mid \varphi \in \Gamma\}$ for $\odot \in \{\boxdot, \Delta\}$.

From an axiomatic point of view, we define the acceptance variant of L, denoted L^{acc}, as follows.

Definition 3. *Axioms and rules of L^{acc} are, besides axioms of L:*

- *Adjunction rule: (Adj)* $\dfrac{\varphi, \quad \psi}{\varphi \wedge \psi}$
- *Restricted Modus Ponens: (r-MP)* $\dfrac{\varphi, \quad \vdash_L \varphi \to \psi}{\psi}$
- *Restricted Δ-necessitation: (r-ΔNec)* $\dfrac{\vdash_L \varphi}{\Delta\varphi}$
- *$\boxdot$-Modus Ponens: ($\boxdot$-MP)* $\dfrac{\varphi, \quad \boxdot\varphi \to \boxdot\psi}{\psi}$

Theorem 1. *Let L be a Δ-core fuzzy logic. Then the system L^{acc} is a sound and complete axiomatization of the finitary $\models^{acc}_{L}$. That is, for any formulas $\varphi_1, ..., \varphi_n, \psi$, it holds*

$$\varphi_1, ..., \varphi_n \models^{acc}_{L} \psi \quad \textit{iff} \quad \varphi_1, ..., \varphi_n \vdash_{L^{acc}} \psi.$$

Finally, the axiomatisation of $\models^{tol}_{L}$ when L is a Δ-core logic is very similar. Indeed, if we define the operator $\boxdot'$ as $\boxdot'\varphi := \Delta(\neg\varphi \to \neg\neg\varphi)$, then the logic L^{tol} resulting from replacing in Definition 3 the rule ($\boxdot$-IR) by the rule

$$(\boxdot'\text{-IR}) \quad \frac{\varphi, \quad \boxdot'(\varphi) \to \boxdot'(\psi)}{\psi},$$

is a sound and complete axiomatisation of $\models^{tol}_{L}$.

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References

1. P. Cintula, C. Noguera. A general framework for mathematical fuzzy logic. In: P. Cintula, P. Hájek, C. Noguera (eds.) *Handbook of Mathematical Fuzzy Logic - Volume 1.* Studies in logic, Mathematical Logic and Foundations, vol 37. College Publications, London, pp. 103-207, 2011.
2. P. Hájek. *Metamathematics of Fuzzy Logic.* Vol. 4 of Trends in Logic - Studia Logica Library. Kluwer, 1998.
3. D. Dubois, H. Prade. Numerical representation of acceptance. In: Proceedings of the 11th Conference on Uncertainty in Artificial Intelligence (UAI'95), Montreal, pp. 149-156, 1995.
4. J. Gispert, L. Godo, F. Esteva. Non-falsity and general threshold-preserving companions of MTL logics. Int. J. Approx. Reason. 196: 109699, 2026.