Algebraic structure of OFWA operators*

Roberto G. Aragón, Jesús Medina, and Samuel Molina-Ruiz

Department of Mathematics, University of Cádiz, Spain
{roberto.aragon,jesus.medina,samuel.molina}@uca.es

Abstract. This work presents the results of the paper *"OFWA operators as a computational mechanism to aggregate variables", TWMS Journal of Pure and Applied Mathematics (2026), in press.*

Keywords:
Information aggregation, ordered functional weighted averaging operator, complete lattice

1 Introduction

Aggregation operators are fundamental tools in a wide variety of disciplines, including fuzzy logic, data analysis, and multi-criteria decision making [1, 8, 9]. Their purpose is to combine several numerical inputs into a single representative value while preserving meaningful information about the original data.

Ordered Weighted Averaging (OWA) operators, introduced by Yager [12,13], have been successfully applied in many contexts due to their ability to model different aggregation attitudes, such as optimism or pessimism. Despite their versatility, an OWA operators always aggregate inputs using the same set of weights, independently of the particular values being aggregated.

In many real-world applications, however, it is desirable for the aggregation mechanism to adapt to the input values themselves. Although a wide range of generalizations of OWA operators have been proposed [5, 10, 11, 14] since their introduction to address this and related issues, Ordered Functional Weighted Averaging (OFWA) operators [2–4, 7] tacke this limitation in a particularly flexible manner by allowing the weights to depend on the inputs. This paper presents the main concepts underlying OFWA operators and discusses their relevance in aggregation tasks, with illustrative examples showcasing the shortcomings of traditional aggregation operators and advantages of this versatile class of operators.

* Partially supported by the project PID2022-137620NB-I00 funded by MICIU/AEI/10.13039/501100011033 and FEDER, UE, Project FEDER-UCA-2024-A2-04 funded by Programa Operativo FEDER Andalucía 2021-2027 and by Consejería de Universidad, Investigación e Innovación, Junta de Andalucía and DGP_PRED_2024_01455 funded by the Junta de Andalucía/CUII and by FSE+.

2 OWA vs OFWA operators

The theory of aggregation operators has become a very extensive and well-developed area of study. Over the years, a great variety of aggregation operators have been proposed in the literature, each tailored to address a specific application or decision-making task. Among these, one classical aggregation operator that has proven to be particularly influential and widely used is the OWA operator, which was introduced by Yager in [13]. The underlying idea behind OWA operators is based separating the importance of the values from the identity of their sources. Instead of assigning weights to each specific source, as is the case with the weighted mean, the aggregation is performed after ordering the values, so that the weights reflect the relative position of the data within the ordered list rather than their original indices. In this way, the weighting list encodes an aggregation attitude, such as optimism, pessimism, or neutrality, by emphasizing higher, lower, or intermediate values, respectively. This ordered weighting mechanism allows OWA operators to unify a wide range of classical aggregation behaviors, including the maximum, minimum, and arithmetic mean, within a single formal setting.

OWA operators satisfies all of the properties listed above, which makes them especially suitable for information aggregation and decision-making processes. However, it suffers from the same issue as many other operators, mainly that they aggregate all the data in $[0,1]^n$ using a fixed, single criterion. This issue may not be apparent in simple or well-behaved scenarios, but it can greatly influence the outcome of more complex aggregation tasks.

In [7], an aggregation operator was defined piecewise in terms of two OWA operators, each acting on a different region. Motivated by the idea of assigning different weights depending on the values being aggregated, they introduced a new class of operators based on the philosophy of OWA operators that precisely aim at providing aggregation mechanisms that are sensitive to the structure and distribution of the inputs.

Definition 1. *Let $\Omega = (\omega_1, \ldots, \omega_n)$ be a list of weights $\omega_i \colon [0,1]^n \to [0,1]$ such that $\sum_{i=1}^{n} \omega_i(a_1, \ldots, a_n) = 1$, for all $(a_1, \ldots, a_n) \in [0,1]^n$. The* n-ary Ordered Functional Weighted Averaging (OFWA) operator *associated with Ω is the mapping $\mathcal{F}_\Omega \colon [0,1]^n \to [0,1]$ defined as*

$$\mathcal{F}_\Omega(a_1, \ldots, a_n) = \sum_{i=1}^{n} \omega_i(a_1, \ldots, a_n)\ a_{\rho(i)}$$

for all $(a_1, \ldots, a_n) \in [0,1]^n$, where $\rho \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation on the index set satisfying that $a_{\rho(n)} \leq \cdots \leq a_{\rho(1)}$.

Whenever the weights in the definition above are constant, we recover the classical OWA operators, from which it follows that OWA operators form a subclass of OFWA operators. However, the addition of functional weights in the definition of OFWA operators offer a lot of flexibility and allow the user

to take into account more complex or multiple criteria when aggregating the input values. One particularly simple construct consists in defining the weights to be piecewise constant, that is, using a family of OWA operators, each acting on a particular region of $[0,1]^n$. The next toy example will showcase how this construct can be used in the setting of risk detection.

Example 1. Our task now is to aggregate the output of two sensors within an alarm-driven system, that is, the system is designed for risk detection such as a gas leak or overheating. Therefore, we fix a risk threshold $\tau \in (0,1)$ and define the weights $\omega_1, \omega_2 \colon [0,1]^2 \to [0,1]$ as

$$\big(\omega_1, \omega_2\big)(a_1, a_2) = \begin{cases} (w_s, 1-w_s) & \text{if } \max\{a_1, a_2\} < \tau \\ (w_r, 1-w_r) & \text{otherwise} \end{cases}$$

for all $(a_1, a_2) \in [0,1]^2$, with $w_s, w_r \in [0,1]$. The value w_s represents the weight used when the input values are not above the risk threshold, and w_r when they are. Thus, we could define for instance $w_s = 0.35$ and $w_r = 0.8$. In this way, we would obtain that

$$\begin{aligned} \mathcal{F}_{(\omega_1,\omega_2)}(0.7, 0.4) &= 0.7 \cdot 0.35 + 0.4 \cdot 0.65 = 0.505 \\ \mathcal{F}_{(\omega_1,\omega_2)}(0.9, 0.4) &= 0.9 \cdot 0.8 + 0.4 \cdot 0.2 = 0.8 \end{aligned}$$

Clearly, the two aggregated outputs differ substantially, even though the inputs are not too different, which suggests that the system has encountered a risk and can now act accordingly.

OFWA operators share many of the interesting properties of OWA operators, mainly the boundary conditions, boundedness, idempotency and symmetry. However, they are not always monotone. In [7], the authors study this property and state sufficient conditions to ensure monotonicity, both for piecewise constant weights and based on a condition on the weights called ratio-increasing. Nonetheless, the ability to define monotone decreasing or even non-monotone OFWA operators broadens the range of potential applications beyond standard multi-criteria decision-making, allowing the user to model scenarios where monotonicity is not a desirable property. Moreover, in [3] the authors discussed the structure of the set of all OFWA operator and proved that it forms a complete lattice together with the pointwise supremum and infimum. This means that we can obtain new OFWA operators by taking the supremum or infimum of any arbitrary family of OFWA operators. Furthermore, these strategies can be used to aggregate different aggregation criteria, modeled by OWA or OFWA operators, proposed by experts according to conservative, associated with the minimum, or aggressive, associated with the maximum, viewpoints. The conservative point of view should be used in scenarios where caution is essential, and the aggressive viewpoint can be used when a more optimistic assessment is appropriate or risk is encouraged.

In the future, we will continue the study into the monotonicity, and other variants such as directional monotonicity [6], of OFWA operators. Moreover,

we will further look into their complete lattice structure and how to aggregate several OFWA operators beyond the conservative and aggressive viewpoints. Furthermore, we will analyze the effect of the weights to the overall operator and introduce a comparative study between OFWA operators and other well-established aggregation operators.

References

1. Roberto G. Aragón, M. Eugenia Cornejo, Jesús Medina, Juan Moreno-García, and Eloísa Ramírez-Poussa. Decision support system for photovoltaic fault detection avoiding meteorological conditions. *International Journal of Information Technology & Decision Making*, 21(03):911–932, 2022.
2. Roberto G. Aragón, Jesús Medina, and Samuel Molina-Ruiz. OFWA operators as a computational mechanism to aggregate variables. *TWMS Journal of Pure and Applied Mathematics*, 2026. In press.
3. Roberto G. Aragón, Jesús Medina, Samuel Molina-Ruiz, and Ronald R. Yager. On the complete lattice structure of ordered functional weighted averaging operators. *Mathematics*, 13(5), 2025.
4. Radomír Halaš, Radko Mesiar, Jozef Pócs, and Andrea Stupňanová. On representation and some properties of OFWA operators. *Fuzzy Sets and Systems*, 508:109333, 2025.
5. Radomír Halaš, Radko Mesiar, Jozef Pócs, Andrea Stupňanová, and LeSheng Jin. A novel generalization of the OWA and IOWA operators based on weighting vectors. *Information Sciences*, 713:122182, 2025.
6. Cédric Marco-Detchart, Humberto Bustince, Javier Fernández, Radko Mesiar, Julio Lafuente, Edurne Barrenechea, and Jesús María Pintor. Ordered directional monotonicity in the construction of edge detectors. *Fuzzy Sets and Systems*, 421:111–132, 2021.
7. Jesús Medina and Ronald R. Yager. OWA operators with functional weights. *Fuzzy Sets and Systems*, 414:38–56, 2021.
8. Ashu Redhu, Reeta Bhardwaj, Kamal Kumar, and Gagandeep Kaur. (p, q)-quasirung orthopair fuzzy Schweizer-Sklar aggregation operators and their application in multi-attribute decision-making. *Journal of Applied Mathematics and Computing*, 71(1):153–182, 2025.
9. Ubaid Ur Rehman and Tahir Mahmood. Ordering blockchain technology for multinational companies based on yager aggregation operators under bipolar fuzzy information. *Journal of Applied Mathematics and Computing*, 70(6):5949–5978, 2024.
10. Vicenç Torra. The weighted OWA operator. *International Journal of Intelligent Systems*, 12(2):153–166, 1997.
11. Vicenç Torra. Andness directedness for operators of the OWA and WOWA families. *Fuzzy Sets and Systems*, 414:28–37, 2021.
12. Ronald R. Yager. Families of OWA operators. *Fuzzy Sets and Systems*, 59(2):125–148, 1993.
13. Ronald R. Yager. On ordered weighted averaging aggregation operators in multicriteria decisionmaking. *IEEE Transactions on Systems, Man, and Cybernetics*, 18(1):183–190, 1998.
14. Ronald R Yager. Generalized OWA aggregation operators. *Fuzzy optimization and decision making*, 3:93–107, 2004.