The Functional Choquet Integral

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Abstract. In this work we follow the paper [1] to introduce the notion of functional Choquet integral (FCI), a new family of fusion functions obtained by replacing the capacity values in the discrete Choquet integral with data-dependent functions applied to the ordered tail of the input vector. Unlike previous extensions, different functions are allowed at each summation term, providing additional modelling flexibility while preserving the ordered-difference structure of the Choquet integral.

Keywords: Fusion function · Choquet integral· Aggregation function.

1 Introduction

In this work we discuss the concept of functional Choquet integrals introduced in [1]. To define these functions, we replace the fuzzy measures $\mu(\{\sigma(i), \ldots, \sigma(n)\})$ which appear in the usual definition of the Choquet integral [3] by a value $F_{n+1-i}(t_{\sigma(i)}, \ldots, t_{\sigma(n)})$ given in terms of a family of functions which take into account only a subset of the inputs.

It is important to remark that, contrary to the case of [2], the used functions may be different from one term to another. Furthermore, no a priori conditions

are imposed on the functions F_{n+1-i}, so we can get a great deal of flexibility. To get further simplification, we restrict our study to fusion functions that depend only on some of the greatest components in the considered input, following the same philosophy as the usual definition of the discrete Choquet integral in terms of the fuzzy measure.

In the following, we introduce the new notion of Functional Choquet integral and review some of its main properties studied in [1].

2 The functional Choquet integral

Let $\mathcal{F} = (F_1, \ldots, F_n)$ be a vector of fusion functions $F_i : [0,1]^i \to [0,1]$, that is, each F_i has exactly i arguments.

Definition 1. *The* functional Choquet integral *(*FCI*, for short) associated to the family $\mathcal{F}$ is the mapping $C_{\mathcal{F}} : [0,1]^n \to [0,+\infty[$ defined, for each $\mathbf{t} = (t_1, \ldots, t_n) \in [0,1]^n$, by:*

$$C_{\mathcal{F}}(\mathbf{t}) = \sum_{i=1}^{n} (t_{\sigma(i)} - t_{\sigma(i-1)}) \cdot F_{n+1-i}(t_{\sigma(i)}, \ldots, t_{\sigma(n)}), \tag{1}$$

where $\sigma : \langle n \rangle \to \langle n \rangle$ is an increasing permutation of the inputs, that is, $t_{\sigma(0)} \leq t_{\sigma(1)} \leq \ldots \leq t_{\sigma(n)}$, with the convention that $t_{\sigma(0)} = 0$.

As a first example consider the case when the functions of the vector $\mathcal{F}$ are constant on their respective domains. For instance, suppose that $F_i \equiv a_i$ for all $i \in \langle n \rangle$, where $a_i \in [0,1]$. Then, for all $(t_1, t_2, \ldots, t_n) \in [0,1]^n$,

$$C_{\mathcal{F}}(t_1, t_2, \ldots, t_n) = \sum_{i=1}^{n} a_{n+1-i} \, (t_{\sigma(i)} - t_{\sigma(i-1)})$$
$$= a_n \, t_{\sigma(1)} + a_{n-1} \, (t_{\sigma(2)} - t_{\sigma(1)}) + \ldots + a_2 \, (t_{\sigma(n-1)} - t_{\sigma(n-2)}) + a_1 \, (t_{\sigma(n)} - t_{\sigma(n-1)}).$$

Then:

- If $F_i \equiv 1$ for all $i \in \langle n \rangle$ (that is, $\mathcal{F} \equiv \mathbf{1}_n$), then $C_{\mathcal{F}}(t_1, t_2, \ldots, t_n) = \max\{t_1, t_2, \ldots, t_n\}$ for all $(t_1, t_2, \ldots, t_n) \in [0,1]^n$.
- If $F_i \equiv 0$ for all $i \in \langle n-1 \rangle$ and $F_n \equiv 1$ (that is, $\mathcal{F} \equiv (0, 0, \ldots, 0, 1)$), then $C_{\mathcal{F}}(t_1, t_2, \ldots, t_n) = \min\{t_1, t_2, \ldots, t_n\}$ for all $(t_1, t_2, \ldots, t_n) \in [0,1]^n$.
- If $F_i \equiv a_i$ for all $i \in \langle n \rangle$ and $1 = a_n \geq a_{n-1} \geq \ldots \geq a_2 \geq a_1 \geq a_0 = 0$, then $C_{\mathcal{F}}$ is the OWA operator associated to the weighting vector $(\omega_1, \omega_2, \ldots, \omega_n) \in [0,1]^n$ defined by $\omega_i = a_{n+1-i} - a_{n-i}$ for all $i \in \langle n \rangle$.

3 Some properties of the Functional Choquet Integral

3.1 Idempotency and Homogeneity

Proposition 1. *The FCI $C_{\mathcal{F}}$ associated to the vector $\mathcal{F} = (F_1, F_2, \ldots, F_n)$ is idempotent if, and only if, $F_n(\mathbf{t}_n) = 1$ for all $t \in (0,1]$.*

In particular, we have the following result.

Corollary 1. *The FCI $C_{\mathcal{F}}$ associated to the vector $\mathcal{F} = (F_1, F_2, \ldots, F_n)$ always satisfies $C_{\mathcal{F}}(\mathbf{0}_n) = 0$, and it verifies the boundary condition $C_{\mathcal{F}}(\mathbf{1}_n) = 1$ if, and only if, $F_n(\mathbf{1}_n) = 1$.*

Regarding homogeneity, we have the following result.

Theorem 1. *The FCI $C_{\mathcal{F}}$ associated to the vector $\mathcal{F} = (F_1, F_2, \ldots, F_n)$ is 1-homogeneous if, and only if, each function F_i is 0-homogeneous.*

3.2 Monotonicity

In order to study monotonicity, we first introduce the following definition.

Definition 2. *A vector of functions $\mathcal{F} = (F_1, F_2, \ldots, F_n)$, where $F_i : \Delta_i \to [0,1]$ for each $i \in \langle n \rangle$, is* increasing *if:*

$$F_n(t_n, t_{n-1}, \ldots, t_2, t_1) \geq F_{n-1}(t_{n-1}, \ldots, t_2, t_1) \geq \ldots \geq F_2(t_2, t_1) \geq F_1(t_1) \quad (2)$$

for each $t_1, t_2, \ldots, t_n \in (0,1]$ such that $0 < t_n \leq t_{n-1} \leq \ldots \leq t_2 \leq t_1$.

Then we have the following result.

Lemma 1. *If the FCI $C_{\mathcal{F}} : [0,1]^n \to \mathbb{R}$ is increasing, then the vector of functions $\mathcal{F} = (F_1, F_2, \ldots, F_n)$ is increasing.*

But the increasingnes of the FCI $C_{\mathcal{F}}$ is not enough to guarantee that each function of the vector $\mathcal{F}$ is increasing, as we illustrate in the following example.

Example 1. Suppose that the functions of the vector $\mathcal{F} = (F_1, F_2, \ldots, F_n)$ are given by:

$$F_n(t_1, t_2, \ldots, t_n) = \frac{t_2 \cdot t_3 \cdot \ldots \cdot t_n}{t_1}, \qquad F_i \equiv 0 \text{ on } \Delta_i \text{ for all } i \in \langle n-1 \rangle.$$

Recall that F_n is defined on $\Delta_n = \{\, (t_1, t_2, \ldots, t_n) \in [0,1]^n : 0 < t_1 \leq t_2 \leq \ldots \leq t_n \,\}$, so $t_1 > 0$. Clearly $\mathcal{F}$ is increasing in the sense of Definition 2. Furthermore, this vector gives place to the FCI such that

$$\begin{aligned} C_{\mathcal{F}}(t_1, t_2, \ldots, t_n) &= t_{\sigma(1)} \cdot F_n(t_{\sigma(1)}, t_{\sigma(2)}, \ldots, t_{\sigma(n)}) = t_{\sigma(1)} \cdot \frac{t_{\sigma(2)} \cdot t_{\sigma(3)} \cdot \ldots \cdot t_{\sigma(n)}}{t_{\sigma(1)}} \\ &= t_{\sigma(2)} \cdot t_{\sigma(3)} \cdot \ldots \cdot t_{\sigma(n)}, \end{aligned}$$

which is increasing on each argument. However, the function F_n is strictly decreasing on its first variable. Therefore, increasingness of $C_{\mathcal{F}}$ does not imply coordinatewise increasingness of each F_i.

Regarding the converse, we introduce the notion of double increasingness.

Definition 3. *A vector of functions $\mathcal{F} = (F_1, F_2, \ldots, F_n)$ is* double-increasing *when$\mathcal{F}$ is increasing and, for each $i \in \langle n \rangle$, the function $F_i : \Delta_i \to [0,1]$ is increasing.*

Finally we get the following main result.

Theorem 2. *If the vector $\mathcal{F}$ is double-increasing, then the FCI $C_{\mathcal{F}}$ is increasing.*

Corollary 2. *If the vector $\mathcal{F}$ is double-increasing and $F_n(\mathbf{1}_n) = 1$, then the FCI $C_{\mathcal{F}}$ is an aggregation function.*

Corollary 3. *Let $\{\, \mathcal{A}_i : [0,1]^i \to [0,1] \,\}_{i=1}^n$ be a set of n aggregation functions such that:*

$$\mathcal{A}_n(t_n, t_{n-1}, \ldots, t_2, t_1) \geq \mathcal{A}_{n-1}(t_{n-1}, \ldots, t_2, t_1) \geq \ldots \geq \mathcal{A}_2(t_2, t_1) \geq \mathcal{A}_1(t_1)$$

for each $t_1, t_2, \ldots, t_n \in (0,1]$ such that $0 < t_n \leq t_{n-1} \leq \ldots \leq t_2 \leq t_1$. If we define $F_i = \mathcal{A}_i|_{\Delta_i}$ (i.e., as the restriction of $\mathcal{A}_i$ to Δ_i) for all $i \in \langle n \rangle$, then the FCI $C_{\mathcal{F}}$ is an aggregation function.

4 Conclusions

In this work we have presented the definition of Functional Choquet integral and some of its main properties, as discussed in [1]. As commented in that work, this new family of operators provides a large amount of flexibility for possible applications in deep learning, decision making or classification, among others.

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